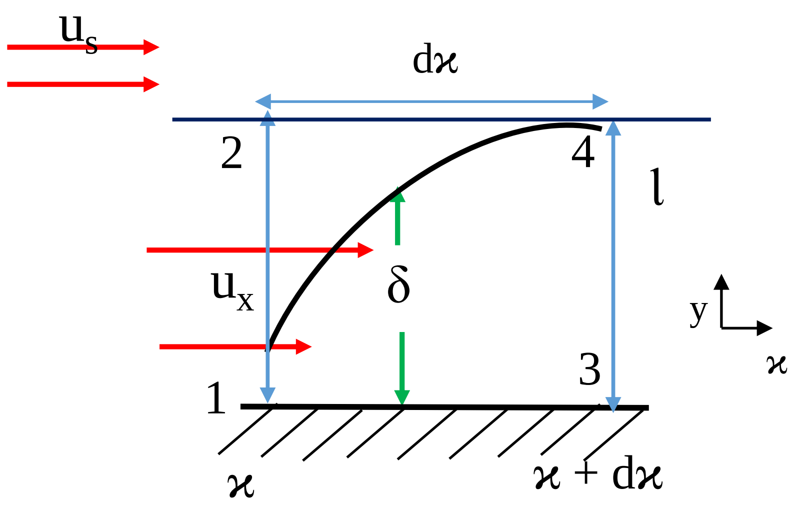
Transfer by molecular Diffusion momentum transfer .

The momentum Transfer equation

For fluid of density (ρ) and viscosity (μ) flow over a *plane surface* and have a velocity of flow (u_s) outside the boundary layer and thickness of boundary layer (δ) forms near the surface and at a height (y) above the surface the velocity is reduced to u_x.



At the equilibrium for the element of fluid boundary the planes (1-2) and (3-4) at distance ***x*** and (ϰ + dϰ) from the leading edge, the (2-4) surface is parallel to (1-3) surface and at distance (l) from it.

When the flow characteristics of the fluid are Newtonian, the **shear stress** (R_y) in a fluid is proportional to the velocity gradient and to the viscosity.

$$\boldsymbol{R_y = -\mu \frac{du_x}{dy}}$$

Where

u_x : is the velocity of the fluid parallel to the surface at distance (y) from it .

R_y : is the shear stress within the fluid , [is a measure of the rate of transfer of momentum per unit area at right angles at the surface].

The negative sign indicates that momentum is transferred from fast to the slow moving fluid and the shear stress acts in such a direction as to oppose the motion of the fluid .

Mass rate of flow mass rate of flow through a strip thickness **dy** at distance **y** from surface = $\boldsymbol{\rho . u_x (1 . dy)}$

Total mass flowrate through plane (1-2) = $\int_0^l \rho \, u_x \, (1 . dy)$

Rate of transfer of momentum through the element strip is the product of the **mass** and **velocity** of an object.

$$= \boldsymbol{\rho . u_x (1 . dy)} \times \boldsymbol{u_x}$$

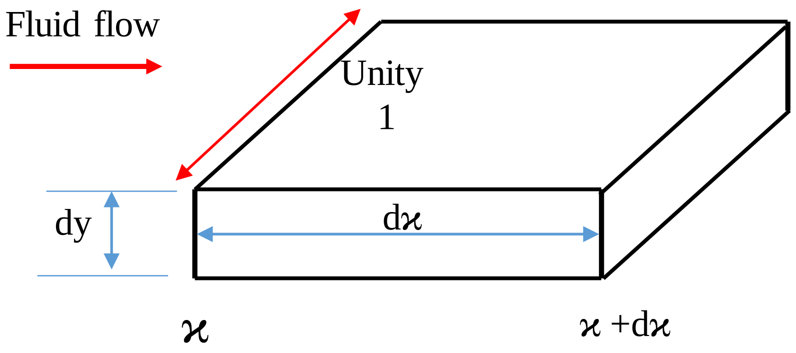
$$= \rho . u_x^2 \, . (1 . dy)$$

Total rate of momentum through plane (1-2) = $\int_0^l \rho \, u_x^2 \, (1 . dy)$

In passing from plane (1-2) to (3-4) the mass flow change by =

$$= \frac{\partial}{\partial x} \int_0^l \rho \, u_x \, (1 . dy) \, dx$$

And rate of momentum change = $\frac{\partial}{\partial x} \int_0^l \rho \, u_x^2 \, (1 . dy) \; dx$



Any differences (increases or decreases) in the mass flow rate of fluid between the flows at the planes (1-2) and (3-4), occur through the plane (2-4). And since plane (2-4) lies outside the boundary layer, the entering fluid must have a velocity (u_s) in direction x.

* The rate of transfer of momentum through (2-4) into the element $= u_s\, [\frac{\partial}{\partial x}\int_0^l \rho\, u_x\, (1.dy)\; dx]$

The net rate of momentum change in x-direction on the element should be equal to momentum added from outside together with net forces acting on it.

The forces in x-direction are:

1- the force of shear resulting from the shear stress (R_o) acting at the surface.

2- the force resulting from a difference in pressure (dP) between planes (1-2) and (3-4).

The net force = $R_o\,(1.\,dx) - \frac{\partial P}{\partial x}\,.\,(dx\,.l) = R_o\, dx - l\,\frac{\partial P}{\partial x}\, dx$

Therefore the net rate of change of momentum in x-direction:

$$\frac{\partial}{\partial x}\int_0^l \rho\, u_x\, (1.dy)\; dx = u_s\, [\frac{\partial}{\partial x}\int_0^l \rho\, u_x\, (1.dy)\; dx] + [R_o\, dx - l\,\frac{\partial P}{\partial x}\, dx]$$

multiply by (-1) and re-arrange

$$u_s\, [\frac{\partial}{\partial x}\int_0^l \rho\, u_x\, (1.dy)\; dx] - \frac{\partial}{\partial x}\int_0^l \rho\, u_x\, (1.dy)\; dx = -R_o\, dx + l\,\frac{\partial P}{\partial x}\, dx \qquad \div dx$$

$$\frac{\partial}{\partial x}\int_0^l \rho\,(u_s - u_x)\,u_x dy \;= -R_o + l\,\frac{\partial P}{\partial x} \qquad (1)$$

The above equation (1) is a general momentum equation which used for :

- Compressible fluid (gasses)
- Incompressible fluid (liquids)
- Also used for turbulent and laminar flow condition

note

Pressure difference can be neglected for liquids $\frac{\partial P}{\partial x} = 0$

but cannot neglected for gasses.

$$\frac{\partial}{\partial x}\int_0^l \rho\,(u_s - u_x)\,u_x dy \;= -R_o \qquad (2)$$

Equation (2) for incompressible fluid

Boundary layer of the streamline

The only forces acting with the fluid are the viscus forces and not transfer of momentum take place by eddy motion. The relation for thickness of the boundary layer has been obtained on the assumption that the velocity profile can be described by polynomial series.

The relation between u_x and y:

$$u_x = u_o + ay + by^2 + cy^3 \quad (3)$$

Where a, b, c are coefficients and u_o can be evaluated " initial velocity"

- At the surface :
- $u_o = 0$
- Shear stress = $R_o = \infty$

$R_o = -\mu \left(\frac{\partial u_x}{\partial y}\right)_{at\ y\ =\ 0}$ at the edge

When y = δ

$u_x = u_s$ and $\frac{\partial u_x}{\partial y} = 0$

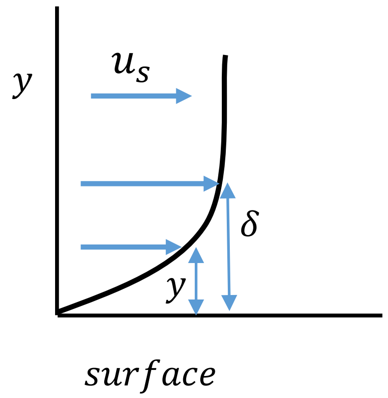
The equation (3) become:

$$u_x = ay + by^2 + cy^3$$

$$\frac{\partial u_x}{\partial y} = a + 2by + 3cy^2$$

$$\frac{\partial^2 u_x}{\partial y^2} = 2b + 6cy$$

at y = 0 , $\frac{\partial^2 u_x}{\partial y^2} = 0$



Velocity profile for streamline

By substitute this condition then we get: b=0

In last equation of (u_x) substitute in u_x for b=0 at y= δ to get

$u_x = a\,\delta + c\,\delta^3$ by differentiate this equation:

$$\frac{\partial u_x}{\partial y} = a + 3c\,\delta^2 = 0$$

so $a = -3c\,\delta^2$ substitute in u_x

$u_x = -3c\,\delta^3 + c\,\delta^3 = -2\,c\,\delta^3$

Substitute =>> $u_s = u_x$

$$c = \frac{-u_s}{2\delta^3} \;,\quad a = \frac{3u_s}{2\delta} \;,\quad b=0$$

The equation for velocity profile is therefore

$$u_x = \frac{3u_s}{2}\frac{y}{\delta} - \frac{u_s}{2}\frac{y^3}{\delta^3}$$ divided by u_s

$$\frac{u_x}{u_s} = \frac{3}{2}\frac{y}{\delta} - \frac{y^3}{2\;\delta^3} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad (4)$$

Applied over range $\boldsymbol{0 < y < \delta}$

The equation (2) which is represent the momentum equation for ***incompressible***:

$$\boldsymbol{\rho \frac{\partial}{\partial x} \int_0^l (u_s - u_x)\, u_x dy = -R_o}$$

Integration can be evaluated for streamline flow in boundary layer by considering the range of y from 0 to l, to get:

$$\rho \frac{\partial}{\partial x} \int_0^l (u_s - u_x)\, u_x dy = \frac{\mathbf{39}}{\mathbf{280}} \boldsymbol{u_s^2\, \delta} \qquad (5)$$

shear stress in laminar boundary layer equal to:

$$\boldsymbol{R_o = -\mu \frac{\partial u_x}{\partial y}} \text{ at y=0}$$

$$u_x = \cancel{u_o} + ay + by^2 + cy^3$$

$$\frac{\boldsymbol{\partial u_x}}{\boldsymbol{\partial y}} = \mathrm{a} + 2\mathrm{by} + 3\mathrm{cy}^2$$

At y=0 then $\boldsymbol{\frac{\partial u_x}{\partial y} = a}$

But $a = \frac{3u_s}{2\delta}$ (from previous slide)

$$\boldsymbol{R_o = -\frac{3}{2} \mu \frac{u_s}{\delta}} \qquad \textbf{(6)}$$

By substitution equations (5 and 6) in equation (2):

$$\boldsymbol{\rho \frac{\partial}{\partial x}\left[\frac{39}{280} u_s^2\, \delta\right] = -[-\frac{3}{2}\mu \frac{u_s}{\delta}]}$$

By integration and re-arrange

$$\delta d\delta = \frac{140}{13}\frac{\mu}{\rho}\frac{1}{u_s}\, dx$$

$$\frac{\delta^2}{2} = \frac{140}{13}\frac{\mu}{\rho}\frac{x}{u_s}$$

At x=0 the δ=0 and at x=x then δ= δ

$$\boldsymbol{\delta = 4.64\sqrt{\frac{\mu\, x}{\rho u_x}}} \qquad (7)$$

Thickness of laminar layer as function of Reynold number can be obtained by multiply and divided by (x)

$$\boldsymbol{\frac{\delta}{x} = 4.64\sqrt{\frac{\mu}{\rho u_x\, x}}} \qquad \boldsymbol{\frac{\delta}{x} = 4.64 \quad Re_x^{-1/2}} \qquad (8)$$

Where the x is the distance of transfer and δ is the thickness of boundary layer.

Assumptions made:

1- velocity profile is a polynomial series

2- main stream velocity (u_s) is reached at distance (δ) from the surface.

Shear stress at the surface:

From equation (6) : $R_o = -\frac{3}{2}\mu\frac{u_s}{\delta}$

$$\boldsymbol{R_o = -\frac{3}{2}\mu\frac{u_s}{x}\frac{1}{4.64\sqrt{\frac{\rho u_s x}{\mu}}}}$$ *by substitute δ value from equation (8)*

? $\boldsymbol{\rho\ u_s^2 = \mu\frac{u_s}{x}}$ **[you can checking the units)**

$$R_o = -0.323\ \rho\ u_s^2\sqrt{\frac{\mu}{\rho u_x x}}$$

$$R_o = -0.323\ \rho\ u_s^2\ Re^{-1/2}$$

$$\frac{R_o}{\rho u_s^2} = -0.323\ Re^{-1/2} \qquad (9)$$

this equation apply to modify chart between log (R/ρ u_s^2) and log Re, With slope (1/2) and intercept =0.323

Mean friction forces at the surface is multiply by R between x=0 and x= x

$$\frac{R_o}{\rho u_s^2} = \int_0^x \frac{R}{\rho\, u_s^2}\, dx \;\; = 0323\sqrt{\frac{\mu}{\rho u_x\, x}}\, dx = 0.646\, Re^{-1/2}$$ (see $\int \frac{1}{\sqrt{x}} = 2\sqrt{x}$

$$\left(\frac{R_o}{\rho u_s^2}\right)_{mean} = 0.646\, Re^{-1/2} \qquad (10)$$

Total frictional forces {mean friction factor for laminar sublayer}